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String propagators in time-dependent and time-independent homogeneous plane waves

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Abstract: For a special time-dependent homogeneous plane wave background that includes a null singularity we construct the closed string propagators. We carry out the summation over the oscillator modes and extract the worldsheet spacetime structures of string propagators specially near the singularity. We construct the closed string propagators in a time-independent smooth homogeneous plane wave background characterized by the constant dilaton, the constant null NS-NS field strength and the constant magnetic field. By expressing them in terms of the hypergeometric function we reveal the background field dependences and the worldsheet spacetime structures of string propagators. The conformal invariance condition for the constant dilaton plays a role to simplify the expressions of string propagators.

Keywords: Penrose limit and pp-wave background, Bosonic Strings.

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Contents

1. Introduction

Plane wave spacetimes have led to remarkable developments in our understanding of the various aspects of general relativity, string theory and Yang-Mills theory. The interesting plane wave background with maximal supersymmetry [1, 2] has been produced by the Penrose limit [3, 4] on the $AdS_5 \times S^5$ solution of type IIB theory. The string theory in this time-independent background is exactly solvable [5, 6] so that we can observe BMN plane-wave/CFT correspondence beyond the supergravity approximation [7].

The time-independent plane waves have been generalized to the time-dependent plane waves in a way which does not destroy the homogeneity of the metric [8, 9]. There has been an investigation of a string theory in a special time-dependent homogeneous plane wave spacetime with a null singularity, that is obtained by a Penrose limit of some cosmological, Dp-brane and fundamental string backgrounds. In the light-cone gauge the string equation has been solved by using Bessel's function and the quantization of the string theory has been performed [8], where it is argued that strings pass through the null singular point. The string theories in the time-dependent plane wave backgrounds have been previously studied from various view points [10]–[12]. On the other hand from a direct analysis of the Killing equations, the time-dependent homogeneous plane wave metrics specified by a rotation matrix f_{ij} have been presented and classified into two families, the regular one generalizing the Cahen-Wallach metrics with the constant symmetric matrix A_{ij} and the singular one generalizing the above special time-dependent plane wave metric with a null singularity [9]. Under some time-dependent coordinate transformation the former family is changed back into the time-independent metric of regular homogeneous plane wave background specified by a time-dependent or time-independent dilaton, the constant rotation matrix f_{ij}, a constant symmetric matrix k_{ij} and a constant anti-symmetric matrix h_{ij} associated with the null NS-NS three-form field strength [13]. The closed string theory in this time-independent smooth homgeneous plane wave background has been solved in the light-cone gauge and the light-cone hamiltonian as well as the string spectrum has been presented.

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There has been also a study of the closed string theory in the parallelizable pp-wave background with a constant dilaton and a constant null NS-NS three-form field strength [14], where it is shown that the parallelizable pp-wave backgrounds are necessarily homogeneous plane wave backgrounds, and that a large class of homogenous plane waves are parallelizable, stating the necessary conditions. Recently various investigations have been presented with respect to the homogeneous plane wave backgrounds, the parallelizable pp-wave backgrounds and the related background such as the Gödel universe [15]–[22].

In order to obtain more insights of the string theory in the time-dependent homogeneous plane wave background with a null singularity we will construct closed string propagators in this singular background. We will specialize to the string propagators with equal worldsheet time and carry out the string mode summation to examine the behaviors of string propagators near the singularity, which are compared with the flat-like logarithmic behavior. We will study the closed string propagation in the time-independent smooth homogeneous plane wave background. The string propagator will be constructed for the time-independent dilaton case to be expressed in terms of the hypergeometric function. The equal-time string propagator produced from it will be expanded in the short-distance separations where it is illustrated how the leading flat-like logarithmic behavior appears together with correction terms. The string propagator to the worldsheet time direction with the worldsheet space-coordinate fixed will be also analyzed. For these two kinds of propagators the f_{ij} and h_{ij} dependences will be argued.

2. String propagators in the time-dependent singular homogeneous plane wave background

We consider the string motion in the $d+2$ dimensional time-dependent homogeneous plane wave spacetime with the metric

$$ds^2 = 2dudv - \frac{k}{u^2}(x^i)^2 du^2 + (dx^i)^2 \,, \qquad k = \text{const}\,, \tag{2.1}$$

which has an interesting property that there exists a null singularity at $u = 0$, that is, an initial singularity. This plane wave metric is produced by a Penrose limit of the FRW metric [2], and of the near-horizon geometries of Dp-brane backgrounds ($k = \frac{(7-p)(p-3)}{16}$) and the fundamental string background ($k = 3/16$) [2, 23, 24]. In the light-cone gauge $U = 2\alpha' p^u \tau$ the bosonic part of the closed string action is given by

$$S = -\frac{1}{4\pi\alpha'}\int d\tau \int_0^\pi d\sigma \left(\partial^a X^i \partial_a X^j \delta_{ij} + \frac{k}{\tau^2} X_i^2\right), \tag{2.2}$$

whose equation of motion is solved explicitly in terms of the Bessel functions $J_{\nu-1/2}(z)$, $Y_{\nu-1/2}(z)$ as [8, 23]

$$X^i(\sigma,\tau) = x_0^i(\tau) + \frac{i}{2}\sqrt{2\alpha'}\sum_{n=1}^{\infty}\frac{1}{n}\Big[Z(2n\tau)\left(\alpha_n^i e^{2in\sigma} + \tilde{\alpha}_n^i e^{-2in\sigma}\right) - \\ - Z^*(2n\tau)(\alpha_{-n}^i e^{-2in\sigma} + \tilde{\alpha}_{-n}^i e^{2in\sigma})\Big] \tag{2.3}$$

with

$$
\begin{aligned}
Z(2n\tau) &\equiv e^{-i\frac{\pi}{2}\nu}\sqrt{\pi n\tau}\left[J_{\nu-1/2}(2n\tau) - iY_{\nu-1/2}(2n\tau)\right], \qquad \nu \equiv \frac{1}{2}(1+\sqrt{1-4k}), \\
x_0^i(\tau) &= \frac{1}{\sqrt{2\nu-1}}\left(\tilde{x}^i\tau^{1-\nu} + 2\alpha'\tilde{p}^i\tau^{\nu}\right),
\end{aligned} \tag{2.4}
$$

where we restrict ourselves to only the region $\tau > 0$ and the case of $0 < k < 1/4$, corresponding to $1/2 < \nu < 1$. Through a relation for the Bessel functions the canonical commutation relations are shown to be satisfied in the time-independent way although the commutators for the modes take the standard forms

$$
\begin{aligned}
\left[\alpha_n^i, \alpha_m^j\right] &= n\delta^{ij}\delta_{n+m}, \qquad \left[\tilde{\alpha}_n^i, \tilde{\alpha}_m^j\right] = n\delta^{ij}\delta_{n+m}, \qquad \left[\alpha_n^i, \tilde{\alpha}_m^j\right] = 0, \\
\left[\tilde{x}^i, \tilde{p}^j\right] &= i\delta^{ij}.
\end{aligned} \tag{2.5}
$$

Taking account of the contribution of $8-d$ spectator dimensions with zero-mode momenta p_s we have the light-cone hamiltonian

$$
\begin{aligned}
H &= -p_u = \frac{p_s^2}{2p_v} + \frac{1}{\alpha' p_v}\mathcal{H}, \\
\mathcal{H} &= \mathcal{H}_0(\tau) + \frac{1}{2}\sum_{n=1}^{\infty}\left[\Omega_n(\tau)(\alpha_{-n}^i\alpha_n^i + \tilde{\alpha}_{-n}^i\tilde{\alpha}_n^i) - B_n(\tau)\alpha_n^i\tilde{\alpha}_n^i - B_n^*(\tau)\alpha_{-n}^i\tilde{\alpha}_{-n}^i\right],
\end{aligned} \tag{2.6}
$$

where $\mathcal{H}_0(\tau)$ is the zero-mode part and both $\Omega_n(\tau)$ and $B_n(\tau)$ are the involved functions expressed in terms of the Bessel functions. This hamiltonian includes the non-diagonal terms specified by $B_n(\tau), B_n^*(\tau)$. The transformation characterizing a new set of time-dependent string modes $\mathcal{A}_n^i, \tilde{\mathcal{A}}_n^i$ as

$$
\begin{aligned}
\frac{i}{n}\left(Z(2n\tau)\alpha_n^i - Z^*(2n\tau)\tilde{\alpha}_{-n}^i\right) &= \frac{i}{\sqrt{\omega_n}}\left(e^{-2i\omega_n\tau}\mathcal{A}_n^i(\tau) - e^{2i\omega_n\tau}\tilde{\mathcal{A}}_{-n}^i(\tau)\right), \\
\frac{i}{n}\left(\dot{Z}(2n\tau)\alpha_n^i - \dot{Z}^*(2n\tau)\tilde{\alpha}_{-n}^i\right) &= 2\sqrt{\omega_n}\left(e^{-2i\omega_n\tau}\mathcal{A}_n^i(\tau) + e^{2i\omega_n\tau}\tilde{\mathcal{A}}_{-n}^i(\tau)\right)
\end{aligned} \tag{2.7}
$$

with $\omega_n \equiv \omega_n(\tau) = \sqrt{n^2 + k/4\tau^2}$, can diagonalize the hamiltonian into

$$
\mathcal{H} = \mathcal{H}_0(\tau) + \sum_{n=1}^{\infty}\omega_n(\tau)\left[\mathcal{A}_n^{i\dagger}(\tau)\mathcal{A}_n^i(\tau) + \tilde{\mathcal{A}}_n^{i\dagger}(\tau)\tilde{\mathcal{A}}_n^i(\tau)\right] + h(\tau), \tag{2.8}
$$

where $\mathcal{A}_n^{i\dagger} = \mathcal{A}_{-n}^i, \tilde{\mathcal{A}}_n^{i\dagger} = \tilde{\mathcal{A}}_{-n}^{i\dagger}$ and $h(\tau)$ is a normal ordering c-function [8]. This expression resembles the hamiltonian of a free massive 2-d field theory with an effective time-dependent mass $\sqrt{k}/\tau$. Under the transformation the mode expansion of $X^i(\sigma,\tau)$ in (2.3) is changed into

$$
\begin{aligned}
X^i(\sigma,\tau) = x_0^i(\tau) + \frac{i}{2}\sqrt{2\alpha'}\sum_{n=1}^{\infty}\frac{1}{\sqrt{\omega_n(\tau)}}&\left[e^{-2i\omega_n\tau}\left(\mathcal{A}_n^i(\tau)e^{2in\sigma} + \tilde{\mathcal{A}}_n^i(\tau)e^{-2in\sigma}\right) - \right. \\
&\left. - e^{2i\omega_n\tau}\left(\mathcal{A}_{-n}^i(\tau)e^{-2in\sigma} + \tilde{\mathcal{A}}_{-n}^i(\tau)e^{2in\sigma}\right)\right],
\end{aligned} \tag{2.9}
$$

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where $\mathcal{A}_n^i, \tilde{\mathcal{A}}_n^i$ obey the following commutation relations in the time-independent way

$$[\mathcal{A}_n^i(\tau), \mathcal{A}_m^{j\dagger}(\tau)] = \delta_{nm}\delta^{ij}\,, \qquad [\tilde{\mathcal{A}}_n^i(\tau), \tilde{\mathcal{A}}_m^{j\dagger}(\tau)] = \delta_{nm}\delta^{ij}\,, \qquad [\mathcal{A}_n^i(\tau), \tilde{\mathcal{A}}_m^{j\dagger}(\tau)] = 0\,. \quad (2.10)$$

Here we are ready to study the propagator of closed string in the time-dependent background in the restricted region $0 < u < \infty$. We construct a closed string propagator ${}_\alpha\langle X^i(\sigma,\tau)X^j(\sigma',\tau')\rangle_\alpha$ where the vacuum is defined as the Fock space state which is annihilated by $\alpha_n^i, \tilde{\alpha}_n^i$ with $n > 0$. Substituting the mode expansion (2.3) and using the commutators for the modes (2.5) we derive a string propagator for the α vacuum which is a vacuum at $\tau = \infty$

$$\begin{aligned} {}_\alpha\langle X^i(\sigma,\tau)X^j(\sigma',\tau')\rangle_\alpha &= -\delta^{ij}\frac{1}{2\nu-1}\left(\frac{\tau'}{\tau}\right)^{1-\nu} 2i\alpha'\tau + \\ &+ \delta^{ij}\alpha'\sum_{n=1}^{\infty}\frac{1}{n}Z(2n\tau)Z^*(2n\tau')\cos 2n(\sigma-\sigma')\,. \end{aligned} \quad (2.11)$$

In the $\nu = 1$ case corresponding to $k = 0$ the zero mode part becomes $-2i\alpha'\tau$, that is the expression for the flat space. For large τ, $Z(2n\tau)$ is expanded as $Z(2n\tau) \cong e^{-2in\tau}[1+O(\tau^{-1}]$ so that the string propagator in the large τ, τ' region is approximately given by

$${}_\alpha\langle X^i(\sigma,\tau)X^j(\sigma',\tau')\rangle_\alpha \cong -\frac{\alpha'}{2}\delta^{ij}\left[\frac{1}{2\nu-1}\left(\frac{\tau'}{\tau}\right)^{1-\nu}\ln z\bar{z} + \ln\left(1-\frac{z'}{z}\right) + \ln\left(1-\frac{\bar{z}'}{\bar{z}}\right)\right], \quad (2.12)$$

where we have used a notation as $z = e^{2i(\tau-\sigma)}, \bar{z} = e^{2i(\tau+\sigma)}, z' = e^{2i(\tau'-\sigma')}$ and $\bar{z}' = e^{2i(\tau'+\sigma')}$. The non-zero mode part is the same form as the flat-space theory. Now we analyze the behavior of the string propagator in the small τ, τ' region. For simplicity we work at equal worldsheet time and focus on its spatial dependence. The oscillator part in (2.11) with $\tau = \tau'$ is then expressed as $\delta^{ij}\alpha'\pi f(\sigma,\sigma',\tau)$ with

$$f(\sigma,\sigma',\tau) = \tau\sum_{n=1}^{\infty}H^{(2)}_{\nu-1/2}(2n\tau)H^{(1)}_{\nu-1/2}(2n\tau)\cos 2n(\sigma-\sigma')\,, \quad (2.13)$$

where $H^{(1)}_{\nu-1/2}$, $H^{(2)}_{\nu-1/2}$ are the Hankel functions of the first and second kind. For small τ the summation over n modes can be approximately replaced by the following integral

$$f(\sigma,\sigma',\tau) \cong \int_0^\infty dx H^{(2)}_{\nu-1/2}(2x)H^{(1)}_{\nu-1/2}(2x)\cos\frac{2(\sigma-\sigma')x}{\tau}\,. \quad (2.14)$$

Plugging

$$H^{(2)}_{\nu-1/2}(2x) = \frac{e^{i\nu\pi}J_{\nu-1/2}(2x) - iJ_{-(\nu-1/2)}(2x)}{\cos\nu\pi}\,, \qquad H^{(1)}_{\nu-1/2}(2x) = H^{(2)*}_{\nu-1/2}(2x) \quad (2.15)$$

into (2.14), we can perform the integral since we are allowed to use the following formulas owing to $-1/2 < -(\nu-1/2) < 0 < \nu - 1/2$

$$\int_0^\infty dx J_\nu^2(x)\cos 2ax = \begin{cases} \dfrac{1}{\pi}Q_{\nu-1/2}(1-2a^2)\,, & \text{for } 0 < a < 1\,, \\ -\dfrac{1}{\pi}\sin\nu\pi Q_{\nu-1/2}(2a^2-1)\,, & \text{for } 1 < a\,, \end{cases}$$

$$\int_0^\infty dx J_\nu(x)J_{-\nu}(x)\cos 2ax = \begin{cases} \frac{1}{2}P_{\nu-1/2}(2a^2-1)\,, & \text{for } 0<a<1\,, \\ 0\,, & \text{for } 1<a\,, \end{cases} \tag{2.16}$$

where $P_{\nu-1/2}$, $Q_{\nu-1/2}$ are the Legendre functions of the first and second kind, and the first formula holds only for $-1/2<\nu$. For $0<|\sigma-\sigma'|/2\tau<1$ we have

$$f = \frac{1}{\cos^2\nu\pi}\left[\frac{1}{2\pi}(Q_{\nu-1}(-y)+Q_{-\nu}(-y)) - \frac{\sin\nu\pi}{2}P_{\nu-1}(y)\right] \tag{2.17}$$

with $y=\frac{(\sigma-\sigma')^2}{2\tau^2}-1$, which is further simplified to be $f=P_{\nu-1}(y)/2\sin\nu\pi$ with $y<1$, where we have used two relations, $Q_\nu(y)=\pi(\cos\nu\pi P_\nu(y)-P_\nu(-y))/2\sin\nu\pi$ and $P_\nu(y)=P_{-\nu-1}(y)$. In the same way for $1<\frac{|\sigma-\sigma'|}{2\tau}$, f is expressed again in a compact form as $f=P_{\nu-1}(y)/2\sin\nu\pi$ but with $y>1$. Thus the oscillator part of string propagator in the small $\tau=\tau'$ region is represented by a single special function

$$\delta^{ij}\frac{\alpha'\pi}{2\sin\nu\pi}P_{\nu-1}\left(\frac{(\sigma-\sigma')^2}{2\tau^2}-1\right). \tag{2.18}$$

Since the Legendre function of the first kind is expressed in terms of the hypergeometric function, the oscillator part of string propagator is expressed separately according to the regions of variable as

$$\delta^{ij}\frac{\alpha'\pi}{2\sin\nu\pi}F\left(1-\nu,\nu,1,1-\frac{(\sigma-\sigma')^2}{4\tau^2}\right), \qquad \text{for } \tfrac{|\sigma-\sigma'|}{2\tau}<1\,, \tag{2.19}$$

$$\begin{aligned}&\delta^{ij}\frac{\alpha'\sqrt{\pi}}{2\sin\nu\pi}(2y)^{\nu-1}\left[\frac{\Gamma(\nu-1/2)}{\Gamma(\nu)}F\left(\frac{1-\nu}{2},\frac{2-\nu}{2},\frac{3}{2}-\nu,\frac{1}{y^2}\right)+\right. \\ &\qquad\left.+\frac{\Gamma(1/2-\nu)}{\Gamma(1-\nu)}\frac{1}{(2y)^{2\nu-1}}F\left(\frac{1+\nu}{2},\frac{\nu}{2},\frac{1}{2}+\nu,\frac{1}{y^2}\right)\right], \quad \text{for } 1<\tfrac{|\sigma-\sigma'|}{2\tau}\,.\end{aligned} \tag{2.20}$$

From (2.19) whose hypergeometric function parametrized by $F(\alpha,\beta,\gamma,z)$ satisfies $\alpha+\beta=\gamma$, we can read the short-distance behavior for $\frac{|\sigma-\sigma'|}{2\tau}\ll 1$

$$\delta^{ij}\alpha'\ln\frac{2\tau}{|\sigma-\sigma'|}\,, \tag{2.21}$$

while (2.20) yields the leading large-distance behavior for $1\ll\frac{|\sigma-\sigma'|}{2\tau}$

$$\delta^{ij}\alpha'\frac{\Gamma(\nu-1/2)\Gamma(1-\nu)}{2\sqrt{\pi}}\left(\frac{\tau}{|\sigma-\sigma'|}\right)^{2(1-\nu)}, \tag{2.22}$$

which shows the power damping, owing to the interval $1/2<\nu<1$. Thus we have observed that the leading short-distance logarithmic behavior of the equal-time string propagator in the small τ region is similar to the string propagator in the flat spacetime.

Now let us consider a closed string propagator ${}_{\mathcal{A}}\langle X^i(\sigma,\tau)X^j(\sigma',\tau)\rangle_{\mathcal{A}}$ with equal world-sheet time where we have chosen the other vacuum as the Fock space state which is annihilated by $\mathcal{A}^i_n,\tilde{\mathcal{A}}^i_n$ with $n>0$, and compare it with the equal-time string propagator for the α vacuum. Using the mode expansion (2.9) together with the commutation relations in (2.10) we obtain an equal-time string propagator for the $\mathcal{A}$ vacuum which is a vacuum at finite τ

$$_{\mathcal{A}}\langle X^i(\sigma,\tau)X^j(\sigma',\tau)\rangle_{\mathcal{A}} = -\delta^{ij}\frac{2i\alpha'\tau}{2\nu-1} + \delta^{ij}\alpha'\sum_{n=1}^{\infty}\frac{1}{\omega_n}\cos 2n(\sigma-\sigma')\,, \tag{2.23}$$

which is rewritten by

$$_{\mathcal{A}}\langle X^i(\sigma,\tau)X^j(\sigma',\tau)\rangle_{\mathcal{A}} = -\delta^{ij}\left(\frac{2i}{\sqrt{1-4k}}+\frac{1}{\sqrt{k}}\right)\alpha'\tau + \delta^{ij}\frac{\alpha'}{2}\sum_{n=-\infty}^{\infty}\frac{1}{\omega_n}e^{2in(\sigma-\sigma')}\,. \tag{2.24}$$

Through the Poisson resummation formula it is possible to perform the mode summation

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\frac{1}{\omega_n}e^{2in(\sigma-\sigma')} &= \sum_{l=-\infty}^{\infty}\int_{-\infty}^{\infty}dxe^{2\pi ixl}\frac{1}{\sqrt{x^2+m^2}}e^{2ix(\sigma-\sigma')} \\ &= 2K_0(2m|\sigma-\sigma'|) + 2\sum_{l\neq 0}K_0(2m|l\pi+\sigma-\sigma'|)\end{aligned} \tag{2.25}$$

with $m=\sqrt{k}/2\tau$, where $K_0(x)$ is the modified Bessel function. The leading short-distance behavior of the string propagator (2.24) for $\frac{\sqrt{k}|\sigma-\sigma'|}{2\tau} \ll 1$ is expressed from the first term in (2.25) as

$$\delta^{ij}\alpha'\ln\frac{2\tau}{\sqrt{k}|\sigma-\sigma'|}\,, \tag{2.26}$$

where we have used a relation between the modified Bessel functions

$$K_0(x) = -I_0(x)\ln\frac{x}{2} + \sum_{k=0}^{\infty}\frac{\psi(k+1)}{(k!)^2}\left(\frac{x}{2}\right)^{2k} \tag{2.27}$$

with the psi function $\psi(x)$ and $I_0(0)=1$. It is interesting that this logarithmic behavior shows the same form as (2.21) that is given in the $\frac{|\sigma-\sigma'|}{2\tau} \ll 1$ region. Thus the propagator (2.24) with (2.25), (2.27) is expanded in $\sqrt{k}|l\pi+\sigma-\sigma'|/\tau$ with integer l for the large τ region. In the large τ limit we recover the flat-like logarithmic propagator as expected from the negligible time-dependent mass. On the other hand we write down the large-distance expansion for $\frac{\sqrt{k}|\sigma-\sigma'|}{2\tau} \gg 1$ that comes from the first term of (2.25) as

$$\delta^{ij}\alpha'\sqrt{\frac{\pi\tau}{2\sqrt{k}|\sigma-\sigma'|}}e^{-\frac{\sqrt{k}}{\tau}|\sigma-\sigma'|}\sum_{n=0}^{\infty}\frac{\Gamma(n+1/2)}{n!\Gamma(-n+1/2)}\left(\frac{\tau}{2\sqrt{k}|\sigma-\sigma'|}\right)^n\,. \tag{2.28}$$

This exponential damping behavior is compared with the power damping one observed in (2.22). For both vacuums the discriminaion between the short distance and the large distance is specified by the similar point $|\sigma-\sigma'| \approx \tau$. Even in the small τ region near the singularity both string propagators can show the flat-like logarithmic behavior in the short space separation restricted by $|\sigma-\sigma'| \ll \tau$. The exponential damping in (2.28) with the time-dependent mass $\sqrt{k}/\tau$ in the equal-time propagator resembles the exponential tail of the positive energy part of the invariant Δ function for the massive scalar field in the space-like separation. In the $\mathcal{A}$ vacuum the mass parameter $m=\sqrt{k}/2\tau$ appears as a special combination of ν and τ, while in the α vacuum the propagator is expressed in terms of the parameter ν. However, we note that there is a similarity that the string propagators in the α vacuum decrease for the large space separation when $\nu<1$, and those in the $\mathcal{A}$ vacuum also decrease when $k>0$ that just corresponds to $\nu<1$.

3. String propagators in the time-independent smooth homogenous plane wave background

We turn to the solvable closed string theory in the time-independent smooth homogeneous plane wave backgrounds with a homogeneous NS-NS three-form field strength and a constant dilaton. We construct a string propagator in the case of four-dimensional plane wave background with metric [13]

$$ds^2 = 2dudv + k_{ij}x^i x^j du^2 + 2f_{ij}x^i dx^j du + (dx^i)^2\,, \tag{3.1}$$

with $i = 1, 2$, where the constant symmetric parameter k_{ij} is given by $k_{ij} = k_i\delta_{ij}$ and the constant antisymmetric rotation parameter f_{ij}, that is interpreted as magnetic field, is taken by $f_{ij} = f\epsilon_{ij}$. This metric is supported by a null 2-form NS-NS potential, $B_{ui} = h_{ij}x^j$ with $h_{ij} = h\epsilon_{ij}$ and a dilaton. For the constant dilaton the string sigma model conformal invariance condition provides

$$k_1 + k_2 = 2f^2 - 2h^2\,. \tag{3.2}$$

The orthogonal gauge for the worldsheet metric gives the sigma model lagrangian $L = \frac{1}{4\pi\alpha'}(g + B)_{MN}\partial_+X^M\partial_-X^N$ with $\alpha' = \frac{1}{2}$ where the string embedding coordinates are given by $X^M = (U, V, X^i)$. The linear but non-diagonal equations for string transverse coordinates X^i were solved in the light-cone gauge by using the frequency base ansatz and analyzing a matrix

$$M(\omega, n) = \begin{pmatrix} \omega^2 + k_1 - 4n^2 & 2if\omega + 4inh \\ -2if\omega - 4inh & \omega^2 + k_2 - 4n^2 \end{pmatrix}. \tag{3.3}$$

The solution for the non-zero mode in the expansion of $X^i = \sum_{n=-\infty}^{\infty} X^i_n(\tau)e^{2in\sigma}$ with $X^i_n = (X^i_{-n})^*$ was presented as

$$X^i_n(\tau) = (-1)^i \sum_{J=1}^{4} \xi^{(n)}_J m_{1i}(\omega^{(n)}_J)e^{i\omega^{(n)}_J\tau}\,, \tag{3.4}$$

where the frequencies $\omega^{(n)}_J (J = 1, \ldots, 4)$ are roots of

$$\omega^4 + (k_1 + k_2 - 4f^2 - 8n^2)\omega^2 - 16nfh\omega + (k_1 - 4n^2)(k_2 - 4n^2) - 16n^2h^2 = 0\,, \tag{3.5}$$

which is given from the constraint det $M = 0$. The matrix $m_{ij}(\omega^{(n)}_J)$ is defined as the minor m_{ij} of $M(\omega, n)$ evaluated for $\omega = \omega^{(n)}_J$. For the zero mode the solution was also derived as

$$X^i_0(\tau) = (-1)^i \sum_{j=1}^{2}[\xi^+_j m_{1i}(\omega_j)e^{i\omega_j\tau} + \xi^-_j m_{i1}(\omega_j)e^{-i\omega_j\tau}]\,, \tag{3.6}$$

where $(\xi^+_j)^\dagger = \xi^-_j$ and the pair frequencies $\{\omega_j, -\omega_j\}$ are roots of (3.5) with $n = 0$. In order to obey the time-independent canonical commutation relations the operators $\xi^{(n)}_J, \xi^\pm_j$ must satisfy

$$\left[\xi^{(-n)}_J, \xi^{(n)}_J\right] = C^{(n)}_J \equiv \frac{1}{m_{11}(\omega^{(n)}_J)\prod_{K\neq J}(\omega^{(n)}_J - \omega^{(n)}_K)}, \tag{3.7}$$

$$\left[\xi^-_j, \xi^+_j\right] = C_j \equiv \frac{1}{2m_{11}(\omega_j)\omega_j\prod_{k\neq j}(\omega^2_j - \omega^2_k)} \tag{3.8}$$

and the other commutators vanish. When $k_1 = k_2 = k$, the roots of (3.5) are explicitly obtained by

$$\{\omega_J^{(n)}\} = \{f \pm \sqrt{f^2 + 4n^2 - k + 4hn},\ -f \pm \sqrt{f^2 + 4n^2 - k - 4hn}\}\,. \qquad (3.9)$$

Here by substituting the conformal invariance condition $k = f^2 - h^2$ (3.2) into (3.9), we can obtain simple expressions without square root and assign the frequencies $\omega_J^{(n)}$ as

$$\omega_1^{(n)} = |2n + h| + f\,, \quad \omega_2^{(n)} = |2n - h| - f\,, \quad \omega_3^{(n)} = -|2n - h| - f\,, \quad \omega_4^{(n)} = -|2n + h| + f\,. \qquad (3.10)$$

For the string mode the solution takes the form

$$X^i(\sigma, \tau) = (-1)^i \sum_{n=1}^{\infty} \left[\sum_{J=1}^{4} \xi_J^{(n)} m_{1i}(\omega_J^{(n)}) e^{i\omega_J^{(n)}\tau} e^{2in\sigma} + \sum_{J=1}^{4} \xi_J^{(-n)} m_{1i}(\omega_J^{(-n)}) e^{i\omega_J^{(-n)}\tau} e^{-2in\sigma} \right]. \qquad (3.11)$$

For simplicity we assume that $0 < h < 2,\ 0 < f$. Since there are relations such as $\omega_1^{(-n)} = -\omega_3^{(n)}, \omega_2^{(-n)} = -\omega_4^{(n)}, \omega_3^{(-n)} = -\omega_1^{(n)}, \omega_4^{(-n)} = -\omega_2^{(n)}$ for $n > 0$, we label the frequencies $\omega_J^{(-n)}$ in the last four terms in (3.11) as $\omega_1^{(-n)} = \omega_3^{(-n)}, \omega_2^{(-n)} = \omega_4^{(-n)}, \omega_3^{(-n)} = \omega_1^{(-n)}, \omega_4^{(-n)} = \omega_2^{(-n)}$, that is, the labelling of $J = 1, 2, 3, 4$ is replaced by that of $J' = 3, 4, 1, 2$. Reshuffling the summation over J' we rewrite the last four terms in (3.11) as

$$\sum_{J'=1}^{4} \xi_{J'}^{(-n)} m_{1i}(-\omega_{J'}^{(n)}) e^{-i\omega_{J'}^{(n)}\tau} e^{-2in\sigma}\,, \qquad (3.12)$$

where $\omega_{J'}^{(-n)} = -\omega_{J'}^{(n)}$. From (3.7) every $C_J^{(n)}$ for $n > 0$ is evaluated as

$$C_1^{(n)} = \frac{1}{16(f+h)^2(2n+f)^2(2n+h)}\,, \qquad C_2^{(n)} = \frac{1}{16(f+h)^2(2n-f)^2(2n-h)}\,,$$
$$C_3^{(n)} = -\frac{1}{16(f-h)^2(2n+f)^2(2n-h)}\,, \qquad C_4^{(n)} = -\frac{1}{16(f-h)^2(2n-f)^2(2n+h)}\,, \qquad (3.13)$$

where $m_{11}(\omega) = \omega^2 + f^2 - h^2 - 4n^2$ has been used but here the other components of the minor are written down as $m_{12}(\omega) = -m_{21}(\omega) = -2if\omega - 4inh, m_{22}(\omega) = m_{11}(\omega)$ for convenience. Since $C_1^{(n)} > 0, C_2^{(n)} > 0, C_3^{(n)} < 0, C_4^{(n)} < 0$ for $n > 0$, the operators such as $\xi_1^{(n)}, \xi_2^{(n)}, \xi_3^{(-n)}, \xi_4^{(-n)}$ in the expansion (3.11) with (3.12) are associated with the creation operators, while $\xi_1^{(-n)}, \xi_2^{(-n)}, \xi_3^{(n)}, \xi_4^{(n)}$ are regarded as the annihilation operators up to the normalizations. For the $h > 2$ or the $h < 0$ case, from (3.13) the assignments of the creation operators or the annihilation operators are changed but the analysis remains essentially the same.

Now we construct a closed string propagator $< X^i(\sigma, \tau) X^j(\sigma', \tau') >$ where the vacuum is defined as the Fock space state which is annihilated by $\xi_1^{(-n)}, \xi_2^{(-n)}, \xi_3^{(n)}, \xi_4^{(n)}$ with $n > 0$. The diagonal (1,1) component of the propagator for the string non-zero mode is shown to

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be equal to the (2,2) component as

$$\begin{aligned}\langle X^1(\sigma,\tau)X^1(\sigma',\tau')\rangle &= \langle X^2(\sigma,\tau)X^2(\sigma',\tau')\rangle \\ &= \frac{1}{4}\sum_{n=1}^{\infty}\Bigg[\left(\frac{1}{2n+h}e^{-i(2n+h+f)(\tau-\tau')}+\frac{1}{2n-h}e^{-i(2n-h-f)(\tau-\tau')}\right)\times \\ &\quad\times e^{-2in(\sigma-\sigma')}+ \\ &\quad+\left(\frac{1}{2n-h}e^{-i(2n-h+f)(\tau-\tau')}+\frac{1}{2n+h}e^{-i(2n+h-f)(\tau-\tau')}\right)\times \\ &\quad\times e^{2in(\sigma-\sigma')}\Bigg], \end{aligned} \tag{3.14}$$

where we have observed that suitable cancellations occur to yield the simplified coefficients. In calculating the (2,2) component we have estimated $m_{12}(-\omega_J^{(n)})$ in the following way. As seen in the steps from (3.11) to (3.12), $m_{12}(-\omega_J^{(n)})$ is regarded as a component of the minor m_{ij} for $M(\omega,-n)$ evaluated at $\omega = -\omega_J^{(n)}$. Therefore $m_{12}(-\omega_J^{(n)}) = -2if\omega - 4i(-n)h|_{\omega=-\omega_J^{(n)}} = m_{21}(\omega_J^{(n)})$. The non-diagonal (2,1) component of the propagator is also equal to the (1,2) component except for the sign as

$$\begin{aligned}\langle X^2(\sigma,\tau)X^1(\sigma',\tau')\rangle &= -\langle X^1(\sigma,\tau)X^2(\sigma',\tau')\rangle \\ &= -\frac{i}{4}\sum_{n=1}^{\infty}\Bigg[\left(\frac{1}{2n+h}e^{-i(2n+h+f)(\tau-\tau')}-\frac{1}{2n-h}e^{-i(2n-h-f)(\tau-\tau')}\right)\times \\ &\quad\times e^{-2in(\sigma-\sigma')}+ \\ &\quad+\left(\frac{1}{2n-h}e^{-i(2n-h+f)(\tau-\tau')}-\frac{1}{2n+h}e^{-i(2n+h-f)(\tau-\tau')}\right)\times \\ &\quad\times e^{2in(\sigma-\sigma')}\Bigg]. \end{aligned} \tag{3.15}$$

If we do not impose (3.2) but use (3.9) as the string frequencies for general values of f, h and k, where the dilaton has time-dependence such that the conformal invariance condition is satisfied, then the string propagator has the mode expansion with the fractional and complicated coefficients, compared with the simple forms $\frac{1}{2n\pm h}$ in (3.14), (3.15).

There remains a task to analyze the zero-mode contributions. For the zero mode the equation (3.5) with $n = 0$ gives the roots $\{\pm|f-h|, \pm(f+h)\}$. The choice of $\omega_j, j = 1,2$ is taken as $\omega_1 = -|f-h|, \omega_2 = f+h$. First we consider the case $h > f$ which yields $k < 0$ through (3.2), that corresponds to $0 < k < 1/4$ for the time-dependent singular homogeneous plane wave metric (2.1) owing to the opposite sign of du^2. In this case both C_j are determined from (3.8) as

$$C_1 = -\frac{1}{16(f-h)^2f^2h}, \qquad C_2 = \frac{1}{16(f+h)^2f^2h}, \tag{3.16}$$

whose signs imply that the operators ξ_1^+, ξ_2^- are essentially treated as the annihilation operators, while ξ_1^-, ξ_2^+ as the creation operators. From this prescription the zero-mode contribution to the string propagator is obtained in a matrix form as

$$\langle X_0^i(\tau)X_0^j(\tau')\rangle = \begin{pmatrix} \cos f(\tau-\tau') & \sin f(\tau-\tau') \\ -\sin f(\tau-\tau') & \cos f(\tau-\tau') \end{pmatrix} \frac{e^{-ih(\tau-\tau')}}{2h}. \tag{3.17}$$

If we next consider the case $f > h$ implying $k > 0$, then C_2 remains intact but C_1 becomes positive as $1/16(f-h)^2 f^2 h$. Since both C_1 and C_2 are positive, the operators ξ_1^-, ξ_2^- are now regarded as the annihilation operators while ξ_1^+, ξ_2^+ as the creation operators. Although this slight change occurs, the resultant zero-mode string propagator becomes the same as (3.17).

Here we return to the non-zero mode string propagators, (3.14) and (3.15), which are expressed in terms of the hypergeometric function as

$$
\begin{aligned}
\langle X^i(\sigma,\tau)X^i(\sigma',\tau')\rangle &= \frac{1}{4}\Bigg[\frac{1}{2+h}\Bigg(e^{-i(h+f)(\tau-\tau')}\frac{\bar{z}'}{\bar{z}}F\left(\frac{h}{2}+1,1,\frac{h}{2}+2,\frac{\bar{z}'}{\bar{z}}\right)+ \\
&\qquad + e^{-i(h-f)(\tau-\tau')}\frac{z'}{z}F\left(\frac{h}{2}+1,1,\frac{h}{2}+2,\frac{z'}{z}\right)\Bigg)+ \\
&\quad + \frac{1}{2-h}\Bigg(e^{i(h+f)(\tau-\tau')}\frac{\bar{z}'}{\bar{z}}F\left(-\frac{h}{2}+1,1,-\frac{h}{2}+2,\frac{\bar{z}'}{\bar{z}}\right)+ \\
&\qquad + e^{i(h-f)(\tau-\tau')}\frac{z'}{z}F\left(-\frac{h}{2}+1,1,-\frac{h}{2}+2,\frac{z'}{z}\right)\Bigg)\Bigg] \qquad (3.18)
\end{aligned}
$$

and

$$
\begin{aligned}
\langle X^2(\sigma,\tau)X^1(\sigma',\tau')\rangle &= -\frac{i}{4}\Bigg[\frac{1}{2+h}\Bigg(e^{-i(h+f)(\tau-\tau')}\frac{\bar{z}'}{\bar{z}}F\left(\frac{h}{2}+1,1,\frac{h}{2}+2,\frac{\bar{z}'}{\bar{z}}\right) \\
&\qquad - e^{-i(h-f)(\tau-\tau')}\frac{z'}{z}F\left(\frac{h}{2}+1,1,\frac{h}{2}+2,\frac{z'}{z}\right)\Bigg)- \\
&\quad - \frac{1}{2-h}\Bigg(e^{i(h+f)(\tau-\tau')}\frac{\bar{z}'}{\bar{z}}F\left(-\frac{h}{2}+1,1,-\frac{h}{2}+2,\frac{\bar{z}'}{\bar{z}}\right)- \qquad (3.19) \\
&\qquad - e^{i(h-f)(\tau-\tau')}\frac{z'}{z}F\left(-\frac{h}{2}+1,1,-\frac{h}{2}+2,\frac{z'}{z}\right)\Bigg)\Bigg],
\end{aligned}
$$

where the same notations as in (2.12) have been used. We use the expansion formula of the hypergeometric function $F(\alpha,\beta,\alpha+\beta,z)$ about $z=1$ to express the diagonal string propagator (3.18) with $\tau=\tau'$ as the expansion in $\sin\Delta\sigma$ with $\Delta\sigma=\sigma-\sigma'$

$$
\begin{aligned}
\langle X^i(\sigma,\tau)X^i(\sigma',\tau)\rangle &= F(h)+F(-h)\,, \\
F(h) &= \frac{1}{8}\sum_{n=0}^{\infty}\frac{\Gamma(\frac{h}{2}+1+n)}{\Gamma(\frac{h}{2}+1)\Gamma(n+1)}\times \\
&\quad\times\Bigg[2\left(\psi(n+1)-\psi\left(\frac{h}{2}+1+n\right)\right)\cos\left((n+2)\Delta\sigma-\frac{\pi}{2}n\right)- \\
&\quad - \ln(1-e^{-2i\Delta\sigma})e^{-i((n+2)\Delta\sigma-\frac{\pi}{2}n)}- \\
&\quad - \ln(1-e^{2i\Delta\sigma})e^{i((n+2)\Delta\sigma-\frac{\pi}{2}n)}\Bigg](2\sin\Delta\sigma)^n, \qquad (3.20)
\end{aligned}
$$

where there are no f-dependences. The short-distance behavior at $\sigma\approx\sigma'$ is specified by the leading $n=0$ term that contains the logarithmic function, which is compared with (2.12), (2.21) and (2.26). The off-diagonal string propagator (3.19) with $\tau=\tau'$ is also described

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by the expansion in $\sin\Delta\sigma$

$$\begin{aligned}\langle X^2(\sigma,\tau)X^1(\sigma',\tau)\rangle &= G(h) - G(-h)\,,\\ G(h) &= -\frac{i}{8}\sum_{n=0}^{\infty}\frac{\Gamma(\frac{h}{2}+1+n)}{\Gamma(\frac{h}{2}+1)\Gamma(n+1)}\times\\ &\quad\times\Bigg[-2i\left(\psi(n+1)-\psi\left(\frac{h}{2}+1+n\right)\right)\sin\left((n+2)\Delta\sigma-\frac{\pi}{2}n\right)-\\ &\quad - \ln(1-e^{-2i\Delta\sigma})e^{-i((n+2)\Delta\sigma-\frac{\pi}{2}n)}+\\ &\quad + \ln(1-e^{2i\Delta\sigma})e^{i((n+2)\Delta\sigma-\frac{\pi}{2}n)}\Bigg](2\sin\Delta\sigma)^n\,, \end{aligned}\tag{3.21}$$

which is manifestly real and has also no f-dependences. In the short-distance limit the leading $n=0$ term does not include the logarithmic function.

Alternatively, from the expressions (3.14), (3.15) and (3.17), the diagonal equal-time string propagator is given by

$$\langle X^i(\sigma,\tau)X^i(\sigma',\tau)\rangle = \frac{1}{2h}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{n}{n^2-h^2/4}\cos 2n(\sigma-\sigma') \tag{3.22}$$

and the non-diagonal one also takes the form

$$\langle X^2(\sigma,\tau)X^1(\sigma',\tau)\rangle = \frac{1}{4}\sum_{n=1}^{\infty}\frac{h}{n^2-h^2/4}\sin 2n(\sigma-\sigma')\,. \tag{3.23}$$

The expression for the non-zero mode in (3.22) is suggestively compared with the real expressions of the equal-time string propagators in (2.13) and (2.24) with (2.25). In the small h expansion the diagonal propagator (3.22) is even function of $\sigma-\sigma'$ so that it is approximately expressed as

$$\begin{aligned}&\frac{1}{2h}-\frac{1}{2}\ln(2\sin|\sigma-\sigma'|)+\frac{h^2}{8}\Bigg[\zeta(3)+2(\sigma-\sigma')^2\ln 2|\sigma-\sigma'|-3(\sigma-\sigma')^2-\\ &\qquad -\frac{1}{18}(\sigma-\sigma')^4-\frac{1}{1350}(\sigma-\sigma')^6+\cdots\Bigg]+O(h^4)\,, \end{aligned}\tag{3.24}$$

where $\zeta(x)$ is Riemann's zeta function, while the non-diagonal one (3.23) is also expressed as

$$\frac{h}{2}\left[-(\sigma-\sigma')\ln(2|\sigma-\sigma'|)+(\sigma-\sigma')+\frac{1}{18}(\sigma-\sigma')^3+\cdots\right]+O(h^3)\,, \tag{3.25}$$

which is odd function of $\sigma-\sigma'$. In the short-distance limit $\sigma\to\sigma'$ to the space direction the former also shows the logarithmic singular behavior and the latter vanishes.

In order to focus on the dependence of the string propagator on f we put $\sigma=\sigma'$ in (3.14), (3.15) to obtain in a product form

$$\begin{aligned}\langle X^i(\sigma,\tau)X^j(\sigma,\tau')\rangle &= \begin{pmatrix}\cos f(\tau-\tau') & \sin f(\tau-\tau')\\ -\sin f(\tau-\tau') & \cos f(\tau-\tau')\end{pmatrix} I(h)\,,\\ I(h) &= \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{e^{-i(2n+h)(\tau-\tau')}}{2n+h}+\frac{e^{-i(2n-h)(\tau-\tau')}}{2n-h}\right)\end{aligned}\tag{3.26}$$

for the non-zero mode, where the f-dependent part is separated from the h-dependent one. The zero-mode contribution continues to be described by (3.17). The closed string propagator at $\sigma = \sigma'$ is similar to the open string propagator on the worldsheet boundary at $\sigma = 0$ or $\sigma = \pi$ where the open string attaches to the D-brane. This open string propagator leads to the noncommutativity on the D-brane worldvolume in the presence of a background NS-NS three-form field strength [25]. It is interesting that the rotation matrix with the rotation parameter f appears in the non-zero mode propagator (3.26) as well as the zero-mode propagator (3.17). This appearance is also seen in the simple expression of the string propagator for the non-zero mode in the $f \neq 0, h = 0$ case

$$\langle X^i(\sigma,\tau)X^j(\sigma',\tau')\rangle = -\frac{1}{4}\begin{pmatrix} \cos f(\tau-\tau') & \sin f(\tau-\tau') \\ -\sin f(\tau-\tau') & \cos f(\tau-\tau') \end{pmatrix}\left[\ln\left(1-\frac{\bar{z}'}{\bar{z}}\right) + \ln\left(1-\frac{z'}{z}\right)\right] \tag{3.27}$$

with the same notations as in (2.12). In (3.26) the diagonal part has the same mode summation as the off-diagonal part, which is different from the behaviors of the equal-time propagators, (3.22) and (3.23). The mode summation $I(h)$ is separated into the four kinds of terms as

$$\begin{aligned} I(h) = &\frac{\cos h(\tau-\tau')}{2}\sum_{n=1}^{\infty}\frac{n}{n^2-h^2/4}\cos 2n(\tau-\tau') + \\ &+\frac{\sin h(\tau-\tau')}{4}\sum_{n=1}^{\infty}\frac{h}{n^2-h^2/4}\sin 2n(\tau-\tau') + \\ &+i\left[-\frac{\cos h(\tau-\tau')}{2}\sum_{n=1}^{\infty}\frac{n}{n^2-h^2/4}\sin 2n(\tau-\tau') + \right. \\ &\left. +\frac{\sin h(\tau-\tau')}{4}\sum_{n=1}^{\infty}\frac{h}{n^2-h^2/4}\cos 2n(\tau-\tau')\right]. \end{aligned} \tag{3.28}$$

Compared with the real equal-time string propagator, the string propagator at $\sigma = \sigma'$ includes an imaginary part. The real part can be expanded in h in the same way as (3.22), (3.23), where $\sigma - \sigma'$ is replaced by $\tau - \tau'$. In the imaginary part the mode summation can be carried out to yield a simle form

$$-i\frac{\pi}{4}\left[\epsilon(\tau-\tau') - \frac{2}{h\pi}\sin h(\tau-\tau')\right] \tag{3.29}$$

for a small region $|\tau-\tau'| < \pi$. In the short-time limit $\tau \to \tau'$, $I(h)$ exhibits the logarithmic singular behavior but the off-diagonal propagator vanishes owing to a multiplying factor $\sin f(\tau - \tau')$. In view of (3.23) and (3.26) with (3.17) we note that the non-diagonal propagator with $\tau = \tau'$ is proportional to h, while the non-diagonal propagator with $\sigma = \sigma'$ decreases as the magnetic field parameter f becomes much smaller.

4. Conclusion

We have constructed the closed string propagators in the special time-dependent homogeneous plane wave background with a null singularity which is characterized by a parameter k that is related with ν. We have carried out the mode summation for the equal-time string propagators and presented the simple and analytic expressions in terms of the the Legendre function of the first kind for the α vacuum, and the modified Bessel function for the $\mathcal{A}$ vacuum. For the short separation to the worldsheet space direction, both the equal-time string propagaotors show the same flat-like logarithmic behavior in the leading term. For the large separation the string propagator in the α vacuum decreases as the power damping only when $\nu < 1$, while the $\mathcal{A}$ vacuum yields the exponential damping for $k > 0$. We have observed an interesting coincidence between $\nu < 1$ and $k > 0$. We have demonstrated that as we approach to the singular point more closely, the discriminated short-distance region becomes narrower, where the string propagators show the leading flat-like logarithmic behavior.

In the time-independent smooth homogeneous plane wave backgrounds specified by the constant null NS-NS field strength parameter h, the constant magnetic field parameter f and the constant dilaton, we have constructed the closed string propagator in four dimensions. We have observed that the conformal invariance condition for the constant dilaton makes the string propagator tractable and fairly simplified. The obtained string propagator is described by the hypergeometric function so that we can make use of an expansion formula of it to express the equal-time string propagator as the power expansion in $\sin(\sigma - \sigma')$ with the coefficient specified by h only. We have presented an alternative expression expanded in h with the coefficient specified by $\sigma - \sigma'$. In these two reciprocal expressions the diagonal propagators include the logarithmic term at the leading order in each expansion. Analyzing the closed string propagator at $\sigma = \sigma'$ we have demonstrated that it is expressed by a product of two parts, one part including the f dependence is represented by a single rotation matrix and the other part is expanded in h with the coefficient specified by $\tau - \tau'$. We have shown that the presence of the small but finite NS-NS field strength provides the non-zero contribution to the non-diagonal equal-time string propagator, while the non-zero contribution to the non-diagonal string propagator at $\sigma = \sigma'$ is generated when the magnetic field is turned on. In deriving the equal-time string propagators in the two kinds of homogeneous plane wave backgrounds we have observed that there appears a critical point distinguishing the short-distance region from the large-distance region for the time-dependent singular homogeneous plane wave background, while there is not such a point for the time-independent smooth homogeneous plane wave background.

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